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1991 J. Phys. A: Math. Gen. 24 71

(http://iopscience.iop.org/0305-4470/24/1/018)

J. Phys. A: Math. Gen. 24 (1991) 71-78. Printed in the UK

On the structure of the Green–Christoffel tensor

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Received 2 January 1990

Abstract. In the theory of elastic waves for crystals of hexagonal, tetragonal and rhombic symmetries a pseudonormal vector is introduced by which the Green-Christoffel tensor can be represented in Kelvin form, i.e. as a sum of a diagonal tensor and a dyad. It is shown that this approach provides a number of new simple relations for specific directions in the aforementioned crystals. For crystals of trigonal, monoclinic and triclinic symmetries a more general representation of the Green-Christoffel tensor as a sum of a diagonal tensor and two dyads is suggested.

The basic equation for elastic waves in crystals is [1]

$$(\Lambda - v^2)\boldsymbol{u} = 0 \qquad (\Lambda_{ij} - v^2\delta_{ij})u_j = 0 \tag{1}$$

where $\boldsymbol{u}$ is the medium displacement vector; $\Lambda = (\Lambda_{ab}) = (\lambda_{acbd}n_c n_d)$ is the Green-Christoffel tensor; $\boldsymbol{n} = (n_a)$ is the unit vector of the wave normal; λ_{acbd} is the reduced tensor [1] of crystal elastic constants; v is the phase velocity.

In general (1) is rather complex. Therefore in crystals the specific role is played by those directions $\boldsymbol{n}$ for which its solution can be simplified essentially. First of all it concerns symmetry axes, normals to the crystal symmetry planes and directions lying in these planes.

General equation for the cone of *specific* directions in crystals ($\boldsymbol{un} = 0$; $u_i n_i = 0$) was obtained in [1]:

$$\begin{aligned} &\boldsymbol{n}\Lambda^2[\boldsymbol{n}, \Lambda\boldsymbol{n}] = 0 \\ &n_i(\Lambda^2)_{is}[\boldsymbol{n}, \Lambda\boldsymbol{n}]_s = n_i(\Lambda^2)_{is}e_{spq}n_p(\Lambda\boldsymbol{n})_q \\ &\qquad = n_i(\Lambda^2)_{is}e_{spq}n_p\Lambda_{qr}n_r = 0. \end{aligned} \tag{2}$$

For such directions one of the three isonormal waves is strictly transversal, its displacement vector being parallel to $[\boldsymbol{n}, \Lambda\boldsymbol{n}](\boldsymbol{u}\|[\boldsymbol{n}, \Lambda\boldsymbol{n}])$. Directions $\boldsymbol{n}$, for which $\boldsymbol{u}\|\boldsymbol{n}$, are called longitudinal normals and also lie on the cone (2).

Thus for specific directions $\boldsymbol{n}$ we know the displacement of one of the waves ($\boldsymbol{u}\|\boldsymbol{n}$ or $\boldsymbol{u}\|[\boldsymbol{n}, \Lambda\boldsymbol{n}]$) and consequently the corresponding phase velocity $v^2 = \boldsymbol{n}\Lambda\boldsymbol{n}$ or $v^2 = [\boldsymbol{n}, \Lambda\boldsymbol{n}]\Lambda[\boldsymbol{n}, \Lambda\boldsymbol{n}]/[\boldsymbol{n}, \Lambda\boldsymbol{n}]^2$. Parameters of two other waves can be found simply. Hence the normals $\boldsymbol{n}$ lying on the cone (2) are some kind of base directions with the help of which one can easily obtain essential characteristics of possible properties of elastic waves in any crystal. The more such base directions we know, the more we can say about the propagation of waves in the medium considered.

0305-4470/91/010071+08$03.50

Kelvin [2] (see also [3]) had proposed using generally for tensor Λ the following representation:

$$\Lambda = \begin{bmatrix} \mathscr{D}_1 & \alpha_1\alpha_2 & \alpha_1\alpha_3 \\ \alpha_2\alpha_1 & \mathscr{D}_2 & \alpha_2\alpha_3 \\ \alpha_3\alpha_1 & \alpha_3\alpha_2 & \mathscr{D}_3 \end{bmatrix}. \tag{3}$$

In hexagonal, tetragonal and rhombic crystals (referred to henceforth as HTR-crystals) this representation (3) can be utilized [4]. According to [1] we have for hexagonal crystals

$$\begin{aligned} &\Lambda_{11} = \lambda_{11}n_1^2 + \lambda_{66}n_2^2 + \lambda_{44}n_3^2 && \Lambda_{12} = (\lambda_{11} - \lambda_{66})n_1n_2 \\ &\Lambda_{22} = \lambda_{66}n_1^2 + \lambda_{11}n_2^2 + \lambda_{44}n_3^2 && \Lambda_{13} = (\lambda_{13} + \lambda_{44})n_1n_3 \\ &\Lambda_{33} = \lambda_{44} + (\lambda_{33} - \lambda_{44})n_3^2 && \Lambda_{23} = (\lambda_{13} + \lambda_{44})n_2n_3. \end{aligned} \tag{4}$$

In tetragonal crystals Λ_{ab} are the same as in (4) excluding $\Lambda_{12} = (\lambda_{12} + \lambda_{66})n_1n_2$. In rhombic crystals we have

$$\begin{aligned} &\Lambda_{11} = \lambda_{11}n_1^2 + \lambda_{66}n_2^2 + \lambda_{55}n_3^2 && \Lambda_{12} = (\lambda_{12} + \lambda_{66})n_1n_2 \\ &\Lambda_{22} = \lambda_{66}n_1^2 + \lambda_{22}n_2^2 + \lambda_{44}n_3^2 && \Lambda_{13} = (\lambda_{13} + \lambda_{55})n_1n_3 \\ &\Lambda_{33} = \lambda_{55}n_1^2 + \lambda_{44}n_2^2 + \lambda_{33}n_3^2 && \Lambda_{23} = (\lambda_{23} + \lambda_{44})n_2n_3 \end{aligned} \tag{5}$$

where $\lambda_{\alpha\beta}$ are in a known way related to λ_{acbd} [1].

It is seen from (2)-(5) that according to [4] the tensor Λ can be represented as a sum of a diagonal tensor and a dyad:

$$\begin{aligned} &\Lambda = \mathscr{D} + \boldsymbol{N}\cdot\boldsymbol{N} \\ &\mathscr{D} = \begin{bmatrix} \mathscr{D}_1 & 0 & 0 \\ 0 & \mathscr{D}_2 & 0 \\ 0 & 0 & \mathscr{D}_3 \end{bmatrix} \qquad \boldsymbol{N} = (N_a) = (\beta_1n_1, \beta_2n_2, \beta_3n_3). \end{aligned} \tag{6}$$

Then we have in hexagonal crystals

$$\begin{aligned} &\mathscr{D}_1 = \mathscr{D}_2 = \lambda_{66} + (\lambda_{44} - \lambda_{66})n_3^2 \qquad \mathscr{D}_3 = \lambda_{44} + (\lambda_{33} - \lambda_{44} - \beta_3^2)n_3^2 \\ &\beta_1 = \beta_2 = \sqrt{\lambda_{11} - \lambda_{66}} \qquad \beta_3 = (\lambda_{13} + \lambda_{44})/\sqrt{\lambda_{11} - \lambda_{66}} \end{aligned} \tag{7}$$

in tetragonal crystals

$$\begin{aligned} &\mathscr{D}_1 = \alpha n_1^2 + \lambda_{66}n_2^2 + \lambda_{44}n_3^2 \qquad \mathscr{D}_2 = \lambda_{66}n_1^2 + \alpha n_2^2 + \lambda_{44}n_3^2 \qquad \alpha = \lambda_{11} - \lambda_{12} - \lambda_{66} \\ &\mathscr{D}_3 = \lambda_{44} + (\lambda_{33} - \lambda_{44} - \beta_3^2)n_3^2 \qquad \beta_1 = \beta_2 = \sqrt{\lambda_{12} + \lambda_{66}} \qquad \beta_3 = (\lambda_{13} + \lambda_{44})/\beta_1 \end{aligned} \tag{8}$$

and in rhombic crystals

$$\begin{aligned} &\mathscr{D}_1 = (\lambda_{11} - \beta_1^2)n_1^2 + \lambda_{66}n_2^2 + \lambda_{55}n_3^2 && \beta_1^2 = (\lambda_{12} + \lambda_{66})(\lambda_{13} + \lambda_{55})/(\lambda_{23} + \lambda_{44}) \\ &\mathscr{D}_2 = \lambda_{66}n_1^2 + (\lambda_{22} - \beta_2^2)n_2^2 + \lambda_{44}n_3^2 && \beta_2^2 = (\lambda_{12} + \lambda_{66})(\lambda_{23} + \lambda_{44})/(\lambda_{13} + \lambda_{55}) \\ &\mathscr{D}_3 = \lambda_{55}n_1^2 + \lambda_{44}n_2^2 + (\lambda_{33} - \beta_3^2)n_3^2 && \beta_3^2 = (\lambda_{23} + \lambda_{44})(\lambda_{13} + \lambda_{55})/(\lambda_{12} + \lambda_{66}). \end{aligned} \tag{9}$$

Kelvin [2] and after him Musgrave (3) stated that representation [3] is applicable in the most general case. It was shown, however, [5] that the form (3) is not valid for trigonal, monoclinic and triclinic crystals (TMT-crystals).

Nevertheless another, more general, representation of the Green–Christoffel tensor is possible in TMT-crystals:

$$\Lambda = \mathscr{D} + \boldsymbol{N}\cdot\boldsymbol{N}' + \boldsymbol{N}'\cdot\boldsymbol{N} \qquad \boldsymbol{N} = \alpha\boldsymbol{n} \qquad \boldsymbol{N}' = \alpha'\boldsymbol{n} \tag{10}$$

where $\mathscr{D}$ is diagonal tensor, the components of which are defined by

$$\mathscr{D}_c = \Lambda_{cc} - 2N_cN'_c$$

(there is no summing over c) and α, α' are some constant matrices. In the general case we have

$$\boldsymbol{N} = \begin{bmatrix} \alpha_{11}n_1 + \alpha_{12}n_2 + \alpha_{13}n_3 \\ \alpha_{21}n_1 + \alpha_{22}n_2 + \alpha_{23}n_3 \\ \alpha_{31}n_1 + \alpha_{32}n_2 + \alpha_{33}n_3 \end{bmatrix} \qquad \boldsymbol{N}' = \begin{bmatrix} \alpha'_{11}n_1 + \alpha'_{12}n_2 + \alpha'_{13}n_3 \\ \alpha'_{21}n_1 + \alpha'_{22}n_2 + \alpha'_{23}n_3 \\ \alpha'_{31}n_1 + \alpha'_{32}n_2 + \alpha'_{33}n_3 \end{bmatrix}. \tag{11}$$

Using (10) we obtain

$$\Lambda_{ab} = N_aN'_b + N'_aN_b \qquad a \neq b. \tag{12}$$

According to [1] in TMT-crystals the components Λ_{ab} are:

(i) trigonal crystals

$$\begin{aligned} &\Lambda_{11} = \lambda_{11}n_1^2 + \lambda_{66}n_2^2 + \lambda_{44}n_3^2 + 2\lambda_{14}n_2n_3 && \Lambda_{12} = n_1(an_2 + 2\lambda_{14}n_3) \\ &\Lambda_{22} = \lambda_{66}n_1^2 + \lambda_{11}n_2^2 + \lambda_{44}n_3^2 - 2\lambda_{14}n_2n_3 && \Lambda_{13} = n_1(2\lambda_{14}n_2 + bn_3) \\ &\Lambda_{33} = \lambda_{44} + (\lambda_{33} - \lambda_{44})n_3^2 && \Lambda_{23} = \lambda_{14}(n_1^2 - n_2^2) + bn_2n_3 \\ &a = \lambda_{11} - \lambda_{66} \qquad b = \lambda_{13} + \lambda_{44} \end{aligned} \tag{13}$$

(ii) monoclinic crystals

$$\begin{aligned} &\Lambda_{11} = \lambda_{11}n_1^2 + \lambda_{66}n_2^2 + \lambda_{55}n_3^2 + 2\lambda_{16}n_1n_2 && \Lambda_{12} = \lambda_{16}n_1^2 + \lambda_{26}n_2^2 + a_1n_1n_2 \\ &\Lambda_{22} = \lambda_{66}n_1^2 + \lambda_{22}n_2^2 + \lambda_{44}n_3^2 + 2\lambda_{26}n_1n_2 && \Lambda_{13} = n_3(b_1n_1 + \lambda_{36}n_2) \\ &\Lambda_{33} = \lambda_{55}n_1^2 + \lambda_{44}n_2^2 + \lambda_{33}n_3^2 && \Lambda_{23} = n_3(\lambda_{36}n_1 + c_1n_2) \\ &a_1 = \lambda_{12} + \lambda_{66} \qquad b_1 = \lambda_{13} + \lambda_{55} \qquad c_1 = \lambda_{23} + \lambda_{44} \end{aligned} \tag{14}$$

(iii) triclinic crystals

$$\begin{aligned} &\Lambda_{11} = \lambda_{11}n_1^2 + \lambda_{66}n_2^2 + \lambda_{55}n_3^2 + 2(\lambda_{16}n_1n_2 + \lambda_{15}n_1n_3 + \lambda_{56}n_2n_3) \\ &\Lambda_{22} = \lambda_{66}n_1^2 + \lambda_{22}n_2^2 + \lambda_{44}n_3^2 + 2(\lambda_{26}n_1n_2 + \lambda_{46}n_1n_3 + \lambda_{24}n_2n_3) \\ &\Lambda_{33} = \lambda_{55}n_1^2 + \lambda_{44}n_2^2 + \lambda_{33}n_3^2 \\ &\Lambda_{12} = \lambda_{16}n_1^2 + \lambda_{26}n_2^2 + a_2n_1n_2 + c_2n_1n_3 + b_2n_2n_3 \\ &\Lambda_{13} = \lambda_{15}n_1^2 + \lambda_{46}n_2^2 + c_2n_1n_2 + d_2n_1n_3 + \lambda_{36}n_2n_3 \\ &\Lambda_{23} = \lambda_{56}n_1^2 + \lambda_{24}n_2^2 + b_2n_1n_2 + \lambda_{36}n_1n_3 + g_2n_2n_3 \\ &a_2 = \lambda_{12} + \lambda_{66} \qquad b_2 = \lambda_{25} + \lambda_{46} \\ &c_2 = \lambda_{14} + \lambda_{56} \qquad d_2 = \lambda_{13} + \lambda_{55} \\ &g_2 = \lambda_{23} + \lambda_{44}. \end{aligned} \tag{15}$$

To find vectors $\boldsymbol{N}$, $\boldsymbol{N}'$ it is necessary to utilize values Λ_{ab} from (13)–(15) and to solve the resulting systems of equations (12) with respect to α_{ab}, α'_{ab}.

In the case of trigonal crystals, according to (13), components Λ_{ab} $(a \neq b)$ do not contain elements proportional to n_3^2, therefore vectors $\boldsymbol{N}$ and $\boldsymbol{N}'$ can be chosen in the following way:

$$\boldsymbol{N}=\begin{bmatrix} n_1 \\ xn_1+yn_2 \\ x'n_1+y'n_2 \end{bmatrix} \qquad \boldsymbol{N}'=\begin{bmatrix} \mu n_1 \\ \xi n_1+\eta n_2+\zeta n_3 \\ \xi' n_1+\eta' n_2+\zeta' n_3 \end{bmatrix}. \tag{16}$$

By substituting (13) and (16) into (12) and equating the coefficients at $n_a n_b$ we obtain $\zeta=2\lambda_{14}$, $\zeta'=b$ and for the other nine unknowns—a system of nine equations:

$$\begin{aligned} &\xi+\mu x=0 \qquad \eta+\mu y=a \qquad \xi'+\mu x'=0 \\ &\eta'+\mu y'=2\lambda_{14} \qquad bx+2\lambda_{14}x'=0 \\ &by+2\lambda_{14}y'=b \qquad x\eta'+x'\eta+\xi y'+\xi' y=0 \\ &x\xi'+x'\xi=\lambda_{14} \qquad y\eta'+y'\eta=-\lambda_{14} \end{aligned}$$

the solutions of which are $(p=b/2\lambda_{14},\ \rho=(\lambda_{14}+ap)/(2\lambda_{14}-ap))$

$$\begin{aligned} &y=-\rho\pm\sqrt{\rho^2+\rho} \qquad \mu=-[\lambda_{14}+ap+(2\lambda_{14}-ap)y]/2py(y-1) \\ &x=\pm\lambda_{14}/\sqrt{b\mu} \qquad x'=-px \qquad y'=p(1-y) \qquad \xi=-\mu x \\ &\xi'=p\mu x \qquad \eta=a-\mu y \qquad \eta'=2\lambda_{14}-p\mu(1-y). \end{aligned}$$

All the combinations of signs at x and y give four solutions.

In the case of monoclinic crystals on the base of (14) we utilize for vectors $\boldsymbol{N}$, $\boldsymbol{N}'$ the following representation:

$$\boldsymbol{N}=\begin{bmatrix} \alpha n_1+\beta n_2 \\ \gamma n_1+\delta n_2 \\ n_3 \end{bmatrix} \qquad \boldsymbol{N}'=\begin{bmatrix} \alpha' n_1+\beta' n_2 \\ \gamma' n_1+\delta' n_2 \\ \mu' n_3 \end{bmatrix} \tag{17}$$

and from (12) obtain the equations

$$\begin{aligned} &\mu'\alpha+\alpha'=b_1 \qquad \mu'\beta+\beta'=\lambda_{36} \qquad \mu'\gamma+\gamma'=\lambda_{36} \qquad \mu'\delta+\delta'=c_1 \\ &\alpha\gamma'+\alpha'\gamma=\lambda_{16} \qquad \beta\delta'+\beta'\delta=\lambda_{26} \qquad \alpha\delta'+\alpha'\delta+\beta\gamma'+\beta'\gamma=a_1. \end{aligned} \tag{18}$$

By expressing α', β', γ', δ' through the other variables we obtain three equations with five unknowns:

$$\begin{aligned} &\lambda_{36}\alpha+b_1\gamma-2\mu'\alpha\gamma=\lambda_{16} \qquad \lambda_{36}\delta+c_1\beta-2\mu'\beta\delta=\lambda_{26} \\ &c_1\alpha+b_1\delta+\lambda_{36}(\beta+\gamma)-2\mu'(\alpha\delta+\beta\gamma)=a_1. \end{aligned} \tag{19}$$

After multiplication of all these equations by μ' and introduction of the notation $\mu'\alpha=x$, $\mu'\beta=y$, $\mu'\gamma=z$, $\mu'\delta=u$, we have

$$\begin{aligned} &\lambda_{36}x+b_1z-2xz=\lambda_{16}\mu' \qquad \lambda_{36}u+c_1y-2uy=\lambda_{26}\mu' \\ &c_1x+b_1u+\lambda_{36}(y+z)-2(xu+yz)=a_1\mu'. \end{aligned} \tag{20}$$

Regarding pairs of variables x, y or u, z as given parameters, we obtain a system of three linear equations for u, z, μ' or x, y, μ', respectively. For example, by giving x, y we have

$$\begin{aligned} &(2x-b_1)z+\lambda_{16}\mu'=\lambda_{36}x \qquad (2y-\lambda_{36})u+\lambda_{26}\mu'=c_1y \\ &(2x-b_1)u+(2y-\lambda_{36})z+a_1\mu'=c_1x+\lambda_{36}y. \end{aligned} \tag{21}$$

Thus one can easily express z, u, μ' and all the other unknowns in terms of x and y for which an arbitrary value can be chosen. Therefore in the case of monoclinic crystals we have continuous two-parameter set of pairs of vectors $\boldsymbol{N}$, $\boldsymbol{N}'$ realizing representation (10). One can use the arbitrariness of the parameters x, y or u, z to simplify some calculations. However, this arbitrariness is restricted. Thus we cannot set x and y equal to zero simultaneously because the system becomes homogeneous, and for existence of solutions its determinant should be equal to zero, which means that these must be some algebraic relation between the elastic constants of a crystal. For example, by setting $\mu'=0$ we obtain a linear system of three equations with four variables:

$$\alpha\lambda_{36}+b_1\gamma=\lambda_{16} \qquad \delta\lambda_{36}+c_1\beta=\lambda_{26} \qquad c_1\alpha+b_1\delta+\lambda_{36}(\beta+\gamma)=a_1 \tag{22}$$

which has a continuous one-parameter set of solutions because arbitrary values can be given to one of the variables α, β, γ or δ.

In the case of triclinic crystals, due to the absence of terms with n_3^2 in Λ_{12}, Λ_{13} and Λ_{23}, we can suppose, in general, expressions (15) for $\boldsymbol{N}$: $\alpha_{13}=\alpha_{23}=\alpha_{33}=0$ (or analogously for $\boldsymbol{N}'$). As a result we obtain rather a complex system of 15 equations with 15 unknowns. However, its solution can be obtained numerically with the known values of elastic constants (which seem to not be defined yet for triclinic crystals). It is natural to suppose that equations (12) must have solutions in this case too.

Let us consider applications of the representation obtained for the Green–Christoffel tensor.

According to representation (6) in HTR-crystals the vector $\boldsymbol{N}$ which we call a pseudonormal, corresponds to each wave normal $\boldsymbol{n}$. Pseudonormal $\boldsymbol{N}$ is not a unit vector. Moreover, its length varies according to changing of $\boldsymbol{n}$.

Using the notion of pseudonormals we can define in HTR-crystals (analogously to [1], section 17) a number of additional extracted directions for which the basic equation (1) has simple solutions.

Thus if longitudinal normals ($\boldsymbol{u}\|\boldsymbol{n}$) are defined by the condition $[\boldsymbol{n}, \Lambda\boldsymbol{n}]=0$ then the directions of longitudinal pseudonormals ($\boldsymbol{u}\|\boldsymbol{N}$) are defined by the condition $[\boldsymbol{N}, \Lambda\boldsymbol{N}]=0$, which due to (6) is reduced to $[\boldsymbol{N}, \mathscr{D}\boldsymbol{N}]=0$ and, because of the diagonal nature of $\mathscr{D}$, is more simple. Then equation (2), for the specific direction, cone transforms into an equation for the pseudospecific direction cone ($\boldsymbol{u}\boldsymbol{N}=0$):

$$\boldsymbol{N}\mathscr{D}^2[\mathscr{D}\boldsymbol{N}, \boldsymbol{N}]=0 \qquad (\boldsymbol{u}\|[\boldsymbol{N}, \mathscr{D}\boldsymbol{N}]) \tag{23}$$

where longitudinal pseudonormals also lie. Note that some of the pseudospecific directions may coincide with specific ones ($\boldsymbol{N}\|\boldsymbol{n}$). Phase velocities of pseudo-longitudinal waves ($\boldsymbol{u}\|\boldsymbol{N}$) in the case of longitudinal pseudonormals are

$$v^2=\boldsymbol{N}\Lambda\boldsymbol{N}/\boldsymbol{N}^2=\boldsymbol{N}^2+\boldsymbol{N}\mathscr{D}\boldsymbol{N}/\boldsymbol{N}^2 \qquad ([\boldsymbol{N}, \mathscr{D}\boldsymbol{N}]=0).$$

Velocities of pseudotransversal waves ($\boldsymbol{u}\boldsymbol{N}=0$) for the pseudospecific directions (23) are (cf [1], (12.53))

$$\begin{aligned} v^2&=[\boldsymbol{N}, \mathscr{D}\boldsymbol{N}]\mathscr{D}[\boldsymbol{N}, \mathscr{D}\boldsymbol{N}]/[\boldsymbol{N}, \mathscr{D}\boldsymbol{N}]^2 \\ &=|\mathscr{D}|(\boldsymbol{N}\mathscr{D}\boldsymbol{N}\cdot\boldsymbol{N}\mathscr{D}^{-1}\boldsymbol{N}-\boldsymbol{N}^2)/(\boldsymbol{N}^2\cdot\boldsymbol{N}\mathscr{D}^2\boldsymbol{N}-(\boldsymbol{N}\mathscr{D}\boldsymbol{N})^2). \end{aligned}$$

The condition $[\boldsymbol{N}, \mathscr{D}\boldsymbol{N}]=0$ in HTR-crystals is equivalent to the relations

$$N_1N_2(\mathscr{D}_1-\mathscr{D}_2)=0 \qquad N_1N_3(\mathscr{D}_1-\mathscr{D}_3)=0 \qquad N_2N_3(\mathscr{D}_2-\mathscr{D}_3)=0.$$

In hexagonal crystals $\beta_1=\beta_2$, $\mathscr{D}_1=\mathscr{D}_2$ (see (7)), therefore from $N_3=0$ we have $\boldsymbol{N}\|\boldsymbol{n}$, i.e. longitudinal normals and pseudonormals coincide in this case. The same is true at $N_1=N_2=0$. Different directions for them can be realized only from the condition $\mathscr{D}_1=\mathscr{D}_3$ that gives for longitudinal pseudonormals cone the expression $n_3=\pm\sqrt{(\lambda_{44}-\lambda_{66})/(2\lambda_{44}-\lambda_{66}-\lambda_{33}+\beta_3^2)}$. Accordingly for the cone of the longitudinal normals we have

$$n_3=\pm\sqrt{(\lambda_{11}-\lambda_{13}-2\lambda_{44})/(\lambda_{11}+\lambda_{33}-2\lambda_{13}-4\lambda_{44})}.$$

It is evident that the cones exist only when expressions within the square root are positive and less than unity.

In tetragonal crystals there is also $\beta_1=\beta_2$, and at $N_1=N_2=0$ or $N_3=0$ we have $\boldsymbol{N}\|\boldsymbol{n}$. However, $\mathscr{D}_1\neq\mathscr{D}_2$, therefore only the following conditions remain for longitudinal pseudonormals: (i) $N_1=0$, $\mathscr{D}_2=\mathscr{D}_3$; or (ii) $N_2=0$, $\mathscr{D}_1=\mathscr{D}_3$; or (iii) $\mathscr{D}_1=\mathscr{D}_2=\mathscr{D}_3$. The first two cases give

$$n_3=\pm\sqrt{(\lambda_{11}-\lambda_{12}-\lambda_{44}-\lambda_{66})/(\lambda_{11}-\lambda_{12}+\lambda_{33}-\lambda_{66}-2\lambda_{44}-\beta_3^2)} \tag{24}$$

($n_2=\pm\sqrt{1-n_3^2}$ or $n_1=\pm\sqrt{1-n_3^2}$). In the third case

$$n_3=\pm\left(\frac{\lambda_{11}-\lambda_{12}-2\lambda_{44}}{\lambda_{11}-\lambda_{12}+2\lambda_{33}-4\lambda_{44}-2\beta_3^2}\right)^{1/2} \qquad n_1=\pm n_2=\pm\left(\frac{1-n_3^2}{2}\right)^{1/2}. \tag{25}$$

Of course, the aforementioned restrictions over corresponding expressions are also valid here. For longitudinal normals according to [1, section 38], instead of (24) we have $n_3^2=(\lambda_{11}-\lambda_{13}-2\lambda_{44})/(\lambda_{11}-2\lambda_{13}+\lambda_{33}-4\lambda_{44})$, and instead of (25)

$$n_3^2=(2\lambda_{13}+4\lambda_{44}-\lambda_{11}-\lambda_{12}-2\lambda_{66})/(4\lambda_{13}+8\lambda_{44}-\lambda_{11}-\lambda_{12}-2\lambda_{13}-2\lambda_{66}).$$

In rhombic crystals we have $\boldsymbol{u}\|\boldsymbol{N}$ at $\boldsymbol{N}$ directed along the coordinate axes ($n_1=n_2=0$ etc). Besides that, there are four additional possibilities: $N_1=0$, $\mathscr{D}_2=\mathscr{D}_3$ and two analogous ones, and $\mathscr{D}_1=\mathscr{D}_2=\mathscr{D}_3$.

Consider the cones of specific and pseudospecific directions. In hexagonal crystals all the directions are specific [1, section 32]. Using (23) and (7) one can be easily convinced that it is valid for pseudospecific directions too. In tetragonal crystals [1, (39.10)] specific directions lay in the planes (i) $n_1=0$, (ii) $n_2=0$, (iii) $n_3=0$, (iv) $n_1\pm n_2=0$ and on the surface of the fourth-order cone

$$z^2(Az^2+Br^2)-Cx^2y^2-Dr^4=0 \tag{26}$$

where

$$A=\sigma(\lambda_{33}-\lambda_{11}-\rho'+\varkappa) \qquad B=\rho'\sigma+(\rho'+\varkappa)(\lambda_{33}-\lambda_{13}-2\lambda_{44}-2\sigma)$$
$$C=(\lambda_{11}-\lambda_{12}-\lambda_{66})(\lambda_{13}-\lambda_{12}+\rho') \qquad D=\rho'\varkappa \qquad \rho'=\lambda_{13}-\lambda_{11}+2\lambda_{44}$$
$$\sigma=\lambda_{33}-\lambda_{11}-2\rho',\ \varkappa=\lambda_{66}-\lambda_{44}.$$

As for cone of pseudospecific directions, its equation according to (23) is

$$N_1N_2N_3[\mathscr{D}_1^2(\mathscr{D}_2-\mathscr{D}_3)+\mathscr{D}_2^2(\mathscr{D}_3-\mathscr{D}_1)+\mathscr{D}_3^2(\mathscr{D}_1-\mathscr{D}_2)]=0.$$

Thus coordinate planes $N_1=n_1=0$ etc will also be sheets of the cone of the pseudospecific directions for any HTR-crystals. Besides, these directions are laying on the sixth-order cone

$$\mathscr{D}_1^2(\mathscr{D}_2-\mathscr{D}_3)+\mathscr{D}_2^2(\mathscr{D}_3-\mathscr{D}_1)+\mathscr{D}_3^2(\mathscr{D}_1-\mathscr{D}_2)=(\mathscr{D}_1-\mathscr{D}_2)(\mathscr{D}_2-\mathscr{D}_3)(\mathscr{D}_3-\mathscr{D}_1)=0$$

which decays into three second-order cones:

$$\mathscr{D}_1-\mathscr{D}_2=0 \qquad \mathscr{D}_2-\mathscr{D}_3=0 \qquad \mathscr{D}_1-\mathscr{D}_3=0. \tag{27}$$

In tetragonal crystals the first cone $\mathscr{D}_1-\mathscr{D}_2=0=n_1^2-n_2^2$ coincides with the corresponding cone for specific directions and degenerates into two planes $n_1\pm n_2=0$. However, instead of the fourth-order cone (26) we have for pseudospecific directions two the second-order cones:

$$\mathscr{D}_1-\mathscr{D}_3=\omega x^2-\varkappa y^2+\tau z^2=0 \qquad \mathscr{D}_2-\mathscr{D}_3=\varkappa x^2+\omega y^2+\tau z^2=0$$

$$\omega=\lambda_{11}-\lambda_{12}-\lambda_{66}-\lambda_{44} \qquad \tau=\lambda_{44}-\lambda_{33}+\beta_3^2.$$

For rhombic crystals the cones (27) are

$$\mathscr{D}_2-\mathscr{D}_3=(\lambda_{66}-\lambda_{55})x^2+(\lambda_{22}-\lambda_{44}-\beta_2^2)y^2+(\lambda_{44}-\lambda_{33}+\beta_3^2)z^2=0$$

$$\mathscr{D}_1-\mathscr{D}_3=(\lambda_{11}-\lambda_{55}-\beta_1^2)x^2+(\lambda_{66}-\lambda_{44})y^2+(\lambda_{55}-\lambda_{33}+\beta_3^2)z^2=0$$

$$\mathscr{D}_1-\mathscr{D}_2=(\lambda_{11}-\lambda_{66}-\beta_1^2)x^2+(\lambda_{66}-\lambda_{22}+\beta_2^2)y^2+(\lambda_{55}-\lambda_{44})z^2=0.$$

The equation for the cone of specific directions (2) in rhombic crystals is very cumbersome.

Thus it follows from the foregoing that the treatment of pseudonormals and corresponding relations permits one to expand essentially the set of directions in HTR-crystals for which the basic equation (1) has simple solutions, and to facilitate analysis of the properties of elastic waves propagating there. It should be noted that the relations for pseudonormals $\boldsymbol{N}$ as a rule are more simple than the corresponding relations for normals $\boldsymbol{n}$.

In TMT-crystals, due to their lower symmetry, obtaining solutions to (1) in a simpler form can be done for more a restricted set of wave normal directions. In particular (2) for the ninth-order cone of specific directions in trigonal and monoclinic crystals (and especially in triclinic ones) is more cumbersome than in rhombic crystals. However, by utilizing representation (10) one can obtain some simplifications even in these cases.

For TMT-crystals a special role is played by the directions of the vector $[\boldsymbol{NN}']$. In fact, at $\boldsymbol{u}\|[\boldsymbol{NN}']$ equation (1) is $(\Lambda-v^2)[\boldsymbol{NN}']=(\mathscr{D}-v^2)[\boldsymbol{NN}']=0$. Therefore when all diagonal elements of the tensor $\mathscr{D}$ are equal to each other, i.e. when $\mathscr{D}_1=\mathscr{D}_2=\mathscr{D}_3=v^2$, then the vector $[\boldsymbol{NN}']$ will be an eigenvector of Λ. According to (10), (13) and (16) for the case of trigonal crystals we have

$$\mathscr{D}_1=\Lambda_{11}-2N_1N_1'=(\lambda_{11}-2\mu)n_1^2+\lambda_{66}n_2^2+\lambda_{44}n_3^2+2\lambda_{14}n_2n_3$$

$$\mathscr{D}_2=\Lambda_{22}-2N_2N_2'=\lambda_{66}n_1^2+\lambda_{11}n_2^2+\lambda_{44}n_3^2-2(xn_1+yn_2)(\xi n_1+\eta n_2+\zeta n_3)$$

$$\mathscr{D}_3=\Lambda_{33}-2N_3N_3'=\lambda_{44}+(\lambda_{33}-\lambda_{44})n_3^2-2(x'n_1+y'n_2)(\xi'n_1+\eta'n_2+\zeta'n_3).$$

By equating these expressions to each other we obtain two equations:

$$\begin{aligned}\mathscr{D}_1-\mathscr{D}_2&=An_1^2+Bn_2^2+Cn_3^2+Dn_1n_2+En_1n_3+Fn_2n_3=0\\ \mathscr{D}_2-\mathscr{D}_3&=A'n_1^2+B'n_2^2+C'n_3^2+D'n_1n_2+E'n_1n_3+F'n_2n_3=0\end{aligned}\tag{28}$$

where

$$A=\lambda_{11}-\lambda_{66}-2\mu+2x\xi \qquad A'=\lambda_{66}-\lambda_{44}+2(x'\xi+x\xi') \qquad B=\lambda_{66}-\lambda_{11}+2y\eta$$

$$B'=\lambda_{11}-\lambda_{44}+2(y'\eta'-y\eta) \qquad C=0,\ C'=\lambda_{44}-\lambda_{33} \qquad D=2(x\eta+\xi y)$$

$$D'=2(x'\eta'+\xi'y'-x\eta-\xi y) \qquad E=2x\zeta \qquad E'=2(x'\zeta'-x\zeta)$$

$$F=2(\lambda_{14}+y\zeta) \qquad F'=2(y'\zeta'-y\zeta).$$

So when $\boldsymbol{n}$ lies on the intersection of two second-order cones (28) then $[\boldsymbol{NN}']$ will give direction of displacement of the elastic wave.

This condition is more simple than the condition for the case where the specific direction lies on the ninth-order cone (2). Moreover, the form (10) by condition (28) permits one to obtain the displacement directions and velocities of the three waves (see [6, 7]):

$$\boldsymbol{u}_0 = [\boldsymbol{N}\boldsymbol{N}'] \qquad \boldsymbol{u}_\pm = \sqrt{\boldsymbol{N}'^2}\boldsymbol{N} \pm \sqrt{\boldsymbol{N}^2}\boldsymbol{N}' \qquad v_0^2 = \frac{\boldsymbol{u}_0 \mathscr{D} \boldsymbol{u}_0}{\boldsymbol{u}_0^2}$$

$$v_\pm^2 = v_0^2 \pm (\sqrt{\boldsymbol{N}^2\boldsymbol{N}'^2} \pm \boldsymbol{N}\boldsymbol{N}').$$

Therefore the representation (10) *by condition* (28) permits one to solve at once the main problem of crystal acoustics for the corresponding $\boldsymbol{n}$. It is evident that analogous investigations can be made in the same way for monoclinic crystals using (17)-(22).

Thus one can see that utilizing of representation (10) provides some new possibilities for finding a basic directions of wave normals in TMT-crystals, for which simple solutions exist.

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